

Affine Toda field theory as a 3-dimensional integrable system

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Abstract

The affine Toda field theory is studied as a 2+1-dimensional system. The third dimension appears as the discrete space dimension, corresponding to the simple roots in the A_N affine root system, enumerated according to the cyclic order on the A_N affine Dynkin diagram. We show that there exists a natural discretization of the affine Toda theory, where the equations of motion are invariant with respect to permutations of all discrete coordinates. The discrete evolution operator is constructed explicitly. The thermodynamic Bethe ansatz of the affine Toda system is studied in the limit $L, N \rightarrow \infty$. Some conjectures about the structure of the spectrum of the corresponding discrete models are stated.

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1 Introduction

The Toda field theories were extensively studied as classical and quantum integrable field theories. The latest developments in the study of these models are described in the paper [1]. In the quantum case Toda field theories provide examples of integrable models of quantum field theories with a scalar factorizable S -matrix. The affine Toda field theory of A_N type describes N scalar fields interacting nonlinearly with the Lagrangian

$$L_{AT} = \int \sum_{i=1}^N \left(\frac{1}{2} \left(\frac{\partial \phi_i}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial \phi_i}{\partial x} \right)^2 - \frac{M^2}{\beta^2} \exp(\beta(\phi_i - \phi_{i+1})) \right) dx. \quad (1.1)$$

Here it is assumed that $\phi_{N+1} = \phi_1$.

The mass spectrum and the scattering amplitudes of this model of the quantum field theory were suggested in [3] on the base of the bootstrap principle and the perturbation theory.

The spectrum of the model consists of $N - 1$ massive particles with masses $M_l = M \sin(\pi l/N)$, where M is the renormalized mass and $l = 1, \dots, N - 1$. The scattering amplitude of the l -th particle on the k -th particle is given by the following product

$$S_{k,l} = \prod_{1 \leq n \leq k} \prod_{1 \leq m \leq l} S_{11} \left(\theta + i \frac{\pi(k-1-2n)}{N} - i \frac{\pi(l-1-2m)}{N} \right), \quad (1.2)$$

where

$$S_{11}(\theta) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi}{N}) \sinh(\frac{\theta}{2} - \frac{i\pi}{N} + \frac{ib}{2}) \sinh(\frac{\theta}{2} - \frac{ib}{2})}{\sinh(\frac{\theta}{2} - \frac{i\pi}{N}) \sinh(\frac{\theta}{2} + \frac{i\pi}{N} - \frac{ib}{2}) \sinh(\frac{\theta}{2} + \frac{ib}{2})}.$$

It is interesting to compare the Toda model (1.1) with the so called open Toda field theory, described by the same Lagrangian (1.1), but without the periodicity condition $\phi_{N+1} = \phi_1$. The Liouville theory [2], which describes the 2D-gravity, is a simplest example of such model with $N = 1$. The periodic and open Toda field theories have completely different structure of dynamics for finite N [12, 13]. In particular, the open Toda chain has massless spectrum. Moreover, the open Toda model (with N fields) can be obtained from the model (1.1) in the limit $Z \rightarrow 0$, $M = mZ$, $\phi_i = \tilde{\phi}_i + (i2/\beta) \ln Z$ for $i = 1, \dots, N$, and m being fixed, where the Lagrangian (1.1) becomes the Lagrangian of the open Toda field theory with the mass parameter m and with the fields $\tilde{\phi}_i$, $i = 1, \dots, N$.

In this paper we investigate the possibility of interpreting the affine Toda field theory as the model in $2 + 1$ -dimensions with the discrete second space coordinate $i = 1, \dots, N$. In this interpretation the model (1.1) corresponds to the periodic boundary conditions in the second space dimension, while the open Toda model corresponds to the open boundary conditions. Notice, that if we would not have the first space coordinate x , then the interpretation of the components of the field $\{\phi_i\}$ as an extra space dimension would correspond to the original physical model, suggested by Toda for the description of long molecules [14].

The following results came out of the study of such interpretation of the Toda field theory:

- The discrete local integrable version of the affine Toda field theory is proposed. The equations of motion for certain variables in this discrete model are invariant with respect to any permutation of coordinates. In the classical limit these variables are the τ functions [8, 11]. When the space-time lattice is finite the model provides a finite dimensional approximation to the Toda field theory.

- It is shown that in the limit $N \rightarrow \infty$ the system has a thermodynamical limit. The spectrum of the resulting theory consists of massless scalar particles. The density of the free energy is computed using the thermodynamical Bethe ansatz.

The paper is organized as follows. In section 2 we describe the discrete integrable version of the Toda field theory and show that in the classical and continuous limit it is reduced to the classical Toda field theory (1.1). In the next section, we construct explicitly the evolution operator and, using the discrete Lax operators [4], describe the integrals of motion. The construction is very similar to the one, used for the discrete Sine-Gordon model, see [5, 6, 7]. In section 4 we study this discrete system as a 3-dimensional discrete field theory. In appropriate variables, this provides a system of difference equations for the operators in the extended algebra of observables, which is invariant with respect to permutations of coordinates. In the classical case this system first appeared in [8, 11]. Section 5 contains the analysis of the thermodynamical limit of the affine Toda system, regarded as a 3-dimensional field theory. It is shown that there exists a 3-dimensional thermodynamical limit in which the excitations are massless scalar particles.

2 The discrete space-time Toda field theory

2.1

The classical Toda field theories in the continuous space-time are infinite dimensional integrable Hamiltonian systems. The Lax representation for these systems is well known [12]. In this case the Lax representation corresponds to certain choice of the symplectic leaf in the appropriate Lie bialgebra [15, 16]. This is equivalent to the fact that the Poisson brackets between matrix elements of the Lax operators are given by the so called r -matrix Poisson brackets [17].

If one wants to find an integrable discrete space-time analog of the Toda field theory, the natural way to proceed is to find the appropriate discrete Lax pair [4]. A discrete Hamiltonian system with the Lax representation is usually integrable if the Lax operator has the r -matrix Poisson brackets. In more geometrical language this means that the Lax operator describes certain symplectic leaf in the Poisson Lie group $\widehat{SL}(N)$. The factorization map, constructed in [19, 20, 21] provides a Poisson map from the Poisson Lie group to itself, which, being restricted to the corresponding symplectic leaf, gives the evolution with equations of motion for the coordinate functions, given by the discrete Lax equation.

The discretization on the classical level reduces the infinite dimensional integrable Hamiltonian system to the finite dimensional one. The next step is the quantization.

The quantization replaces the Poisson evolution map by an automorphism of the quantum algebra of observables. Since the classical system is integrable we certainly want to construct an integrable quantization (with extra integrals of motion). If the quantum system can be described in terms of differential operators on an n -dimensional manifold, there should be n such integrals. It is known that if the system admits the quantum Lax representation, and if the quantum Lax operators have the so called R -matrix commutation relations, then one can construct the integrals of such evolution, using the traces of appropriate products of quantum Lax operators.

In the algebraic language, the quantum Lax operator is the universal R -matrix of certain factorizable Hopf algebra evaluated on the tensor product of two representations. For the details see [21].

Now we will describe the quantum discrete Toda system and then show that in the classical limit it gives the discrete version of the Toda field theory. The classical equations of motion for the discrete Toda field theory first appeared in [8] as bilinear difference equations for the corresponding τ function. The Lax representation for these equations is described in [10] without the Hamiltonian interpretation. The Lax representation together with the Hamiltonian interpretation for the Toda chain with discrete time was found in [9].

2.2

Here we fix some notations, used throughout the paper.

A parameter q is some non-zero complex number, $h \equiv \ln(q)$; L and N are arbitrary positive integers, $F \equiv (L, N)$, the greatest common divisor of L and N , and $J = LN/F$, the smallest common multiple.

For any positive integer m and any integer n we denote by $[n]_m$ the non-negative integer, such that

$$[n]_m = n \pmod{m}, \quad 0 \leq [n]_m < m. \quad (2.1)$$

For example, we write $[x]_m = 0$ instead of $x = 0 \pmod{m}$. The function

$$\varepsilon_m(n) = \frac{m - 1 - 2[n]_m}{m}, \quad (2.2)$$

is periodic in argument n :

$$\varepsilon_m(n + m) = \varepsilon_m(n).$$

It also satisfies the following relations:

$$\begin{aligned} \varepsilon_m(n) + \varepsilon_m(-1 - n) &= 0, \\ \varepsilon_m(n) - \varepsilon_m(n - 1) &= 2(\delta_m(n) - 1/m), \end{aligned}$$

where

$$\delta_m(n) = \begin{cases} 1, & \text{if } [n]_m = 0; \\ 0, & \text{otherwise.} \end{cases} \quad (2.3)$$

Notice, that for any finite n

$$\lim_{m \rightarrow \infty} \varepsilon_m(n) = \varepsilon(n) \equiv \begin{cases} 1, & \text{if } n \geq 0; \\ -1, & \text{otherwise.} \end{cases}$$

2.3

Define the algebra $C_q(N, L)$, generated by invertible elements

$$\chi_i(n), \quad i = 1, \dots, N; \quad n = 1, \dots, 2L,$$

with the following commutation relations

$$\begin{aligned} \chi_i(n)\chi_j(m) &= \chi_j(m)\chi_i(n), \quad [n - m]_2 = 0; \\ \chi_i(n)\chi_j(m) &= q^{\Omega(i-j, (n-m-1)/2)} \chi_j(m)\chi_i(n), \quad [m]_2 - [n]_2 = 1, \end{aligned} \quad (2.4)$$

where

$$\Omega(i, n) = 2\delta_L(n)(\delta_N(i) - \delta_N(i + 1)) + 2\delta_L(n + 1)(\delta_N(i) - \delta_N(i - 1)). \quad (2.5)$$

Consider two maps $\kappa_{\pm}: C_q(N, L) \rightarrow C_q(N, L)$ defined on the generators $\chi_i(n)$ as follows:

$$\kappa_{\pm}: \chi_i(n \mp 1) \mapsto \chi_i(n), \quad [n]_2 = 0;$$

$$\kappa_{\pm}: \chi_i(n \mp 1) \mapsto q^{-2} \frac{1 - \chi_{i-1}(n+1)}{1 - q^{-2} \chi_i^{-1}(n+1)} \frac{1 - \chi_{i+1}(n-1)}{1 - q^{-2} \chi_i^{-1}(n-1)} \chi_i^{-1}(n), \quad [n]_2 = 1. \quad (2.6)$$

Here and below we assume that the subindex i , enumerating generators $\chi_i(n)$, and their argument n are taken modulo N and $2L$, respectively:

$$\chi_{i+N}(n) = \chi_i(n + 2L) = \chi_i(n).$$

One can also say that $\chi_i(n)$ satisfy the ‘periodic boundary conditions’ in i and n .

Proposition 1 *The maps κ_{\pm} in (2.6), extended by linearity to the whole algebra¹ $C_q(N, L)$, determine automorphisms of this algebra.*

For each $i = 1, \dots, N$, $n = 1, \dots, 2L$, define the ‘trajectory’ of the discrete Toda system as the sequence of elements in $C_q(N, L)$:

$$\chi_i(n \pm 1, t + 1) = \kappa_{\pm}(\chi_i(n, t)); \quad \chi_i(n) = \begin{cases} \chi_i(n, 1), & \text{if } [n]_2 = 0; \\ \chi_i(n, 0), & \text{otherwise.} \end{cases} \quad (2.7)$$

Notice that the fields $\chi_i(n, t)$ are defined only in the case $[n + t]_2 = 1$. They satisfy the following equations of motions

$$\begin{aligned} q^{-2} \chi_i(n, t + 1) \chi_i(n, t - 1) &= \chi_i(n + 1, t) \chi_i(n - 1, t) \\ &\times \frac{1 - \chi_{i-1}(n + 1, t)}{1 - q^2 \chi_i(n + 1, t)} \frac{1 - \chi_{i+1}(n - 1, t)}{1 - q^2 \chi_i(n - 1, t)}. \end{aligned} \quad (2.8)$$

These are the discrete space-time Heisenberg equations of motion for the discrete Toda field theory.

Now, let us show how to recover the classical continuous Toda field theory (1.1) from this system. First consider the classical discrete Toda, which corresponds to the limit $q \rightarrow 1$ in the quantum discrete Toda model described above. Let ε be the lattice spacing both in the time and space directions. Define the new field:

$$\varphi_i(n\varepsilon, t\varepsilon) = \frac{1}{\beta} \ln \left(-\frac{\chi_i(n, t)}{(\varepsilon M)^2} \right). \quad (2.9)$$

Substituting this into (2.8), we obtain ($q = 1$):

$$\begin{aligned} &\exp(\beta(\varphi_i(x, t + \varepsilon) + \varphi_i(x, t - \varepsilon) - \varphi_i(x + \varepsilon, t) - \varphi_i(x - \varepsilon, t))) \\ &= \frac{1 + (\varepsilon M)^2 \exp(\beta\varphi_{i-1}(x + \varepsilon, t))}{1 + (\varepsilon M)^2 \exp(\beta\varphi_i(x + \varepsilon, t))} \frac{1 + (\varepsilon M)^2 \exp(\beta\varphi_{i+1}(x - \varepsilon, t))}{1 + (\varepsilon M)^2 \exp(\beta\varphi_i(x - \varepsilon, t))}. \end{aligned}$$

In the limit $\varepsilon \rightarrow 0$ these equations reduce to

$$\frac{\partial^2 \varphi_i}{\partial t^2} - \frac{\partial^2 \varphi_i}{\partial x^2} = \frac{M^2}{\beta} (\exp(\beta\varphi_{i+1}) + \exp(\beta\varphi_{i-1}) - 2 \exp(\beta\varphi_i)),$$

which coincide with the Euler-Lagrange equations for the system (1.1), written for the fields

$$\varphi_i = \phi_i - \phi_{i+1}. \quad (2.10)$$

¹Strictly speaking, the maps κ_{\pm} act from the algebra to its completion, which can be defined in many ways. For example, one can consider Laurent power series in χ , or $\chi - 1$, or one can realize χ as operators in a Hilbert space and use the spectral theorem for the definition of rational functions of an operator. We will ignore these problems when possible.

3 Evolution operator and integrals of motion

In this section first we investigate the nature of the automorphisms κ_{\pm} defined in (2.6). In particular we will see how to realize them as inner automorphisms of some completion of $C_q(N, L)$. Then we construct the elements of the algebra of observables, which are conserved under the evolution (2.8).

3.1

Let us complete the algebra $C_q(N, L)$ by formal power series in $\chi_i(n) - 1$. The result denote as $C'_q(N, L)$. Introduce the elements $\psi_i(n)$, exponentials of which coincide with $\chi_i(n)$:

$$\chi_i(n) = -(\varepsilon M)^2 \exp(2\psi_i(n)), \quad i = 1, \dots, N; \quad n = 1, \dots, 2L. \quad (3.1)$$

In fact, the operators $\psi_i(n)$, up to a multiplicative factor and a redefinition of the arguments, are the quantum analogs of the fields (2.9) at a fixed time.

The following commutation relations hold:

$$[\psi_i(n), \psi_j(m)] = \frac{\hbar}{4} \Omega \left(i - j, \frac{n - m - 1}{2} \right), \quad [m]_2 - [n]_2 = 1;$$

$$[\psi_i(n), \psi_j(m)] = 0, \quad [m]_2 = [n]_2,$$

where $\Omega(i, n)$ is given in (2.5).

The center of this algebra is generated by the elements, corresponding to the eigenvectors of the matrix:

$$\begin{bmatrix} 0 & \Omega(i - j, n - m) \\ -\Omega(j - i, m - n) & 0 \end{bmatrix}$$

with zero eigenvalues. We will use the following part of it.

Proposition 2 *The following elements² lie in the center of $C'_q(N, L)$:*

$$u(n) \equiv \sum_{i=1}^N \psi_i(n), \quad n = 1, \dots, 2L;$$

$$v(2k - (1 \pm 1)/2) \equiv \sum_{i=1}^N \sum_{n=1}^L \psi_i(2n - (1 \pm 1)/2) \delta_F(i + n - k), \quad k = 1, \dots, F. \quad (3.2)$$

Consider the quotient algebra of $C'_q(N, L)$ with the extra relations

$$u(n) = v(k) = 0, \quad n = 1, \dots, 2L; \quad k = 1, \dots, 2F. \quad (3.3)$$

The automorphisms κ_{\pm} can be represented as the compositions

$$\kappa_{\pm} = \sigma \circ \lambda_{\pm}, \quad (3.4)$$

where

$$\lambda_{\pm}(\chi_i(n)) = \begin{cases} (\varepsilon M)^4 \chi_i^{-1}(n \pm 1), & \text{if } [n]_2 = 0; \\ \chi_i(n \pm 1), & \text{otherwise;} \end{cases} \quad (3.5)$$

and

$$\sigma(\chi_i(n)) = \chi_i(n), \quad [n]_2 = 0;$$

²The elements $u(n)$, $v(n)$ are not independent, for example, it is easy to see that $\sum_{n=1}^L u(2n) = \sum_{k=1}^F v(2k)$, $\sum_{n=1}^L u(2n - 1) = \sum_{k=1}^F v(2k - 1)$.

$$\sigma(\chi_i(n)) = q^2(\varepsilon M)^4 \chi_i(n) \frac{1 - q^{-2} \chi_i^{-1}(n+1)}{1 - \chi_{i-1}(n+1)} \frac{1 - q^{-2} \chi_i^{-1}(n-1)}{1 - \chi_{i+1}(n-1)}, \quad [n]_2 = 1. \quad (3.6)$$

To describe explicitly the evolution operator, introduce some auxiliary objects.

Define the function

$$g(x, y) = \frac{1}{2} \sum_{j=1}^{N/F} \varepsilon_J \left(\frac{x-y-1}{2} + jL \right) \varepsilon_N \left(\frac{x+y-1}{2} + jL \right), \quad [x+y]_2 = 1. \quad (3.7)$$

It is symmetric under the reflection

$$g(x, y) = g(-x, -y),$$

and satisfies the difference equation:

$$\begin{aligned} & g(x+1, y) + g(x-1, y) - g(x, y+1) - g(x, y-1) \\ &= 2 \left(\delta_L \left(\frac{x-y}{2} \right) (\delta_N(y) - N^{-1}) - J^{-1} \delta_F \left(\frac{x+y}{2} \right) + (LN)^{-1} \right). \end{aligned}$$

Consider the quadratic form

$$H(\psi) = \frac{1}{2} \sum_{i,j=1}^N \sum_{m,n=1}^L s(i-j, m-n) \psi_i(2m) \psi_j(2n), \quad (3.8)$$

where

$$s(i, n) = g(i+2n-1, i) + g(i+2n+1, i), \quad (3.9)$$

and the following element in $C'_q(N, L)$:

$$U = \exp(H(\psi)/h) \prod_{i=1}^N \prod_{n=1}^L \Psi(\chi_i(2n)), \quad (3.10)$$

with $\Psi(x)$ being some solution to the functional equation:

$$\Psi(xq^{-2}) = (1-x)\Psi(x). \quad (3.11)$$

For example, if $|q| < 1$, the solution to (3.11), regular at $x = 0$, has the form

$$\Psi(x) = (q^2 x; q^2)_\infty \equiv \prod_{n=1}^{\infty} (1 - xq^{2n}).$$

Proposition 3 *The automorphism σ (3.6) is inner one in $C'_q(N, L)$ and is represented by the element U :*

$$U^{-1} \chi_i(n) U = \sigma(\chi_i(n)),$$

provided the central elements (3.2) are chosen as in (3.3).

Notice, that the restrictions (3.3) are not essential. In the case the central elements are given arbitrary numerical values, the forms of automorphisms λ_\pm and σ should be simultaneously changed by appropriate numerical factors, keeping the compositions (3.4) fixed.

The automorphisms κ_\pm correspond to time evolution combined with space translations, while their composition

$$\kappa \equiv \kappa_+ \circ \kappa_- = \kappa_- \circ \kappa_+, \quad (3.12)$$

corresponds to pure time translation:

$$\chi_i(n, t+2) = \kappa(\chi_i(n, t)) = (VU)^{-1} \lambda(\chi_i(n, t)) VU = (\varepsilon M)^4 (VU)^{-1} \chi_i^{-1}(n, t) VU, \quad (3.13)$$

where

$$\begin{aligned} V &\equiv \lambda_+(U) = \lambda_-(U), \\ \lambda &\equiv \lambda_+ \circ \lambda_- = \lambda_- \circ \lambda_+. \end{aligned}$$

Note, that automorphisms λ_{\pm} themselves are also inner ones in $C'_q(N, L)$, but their role is auxiliary in the sense that in the ‘two-step’ time evolution (3.13) they combine into the involution λ .

3.2

Here we describe some algebra which will be used in the description of quantum integrals for the evolution (2.8).

Denote by $A_q(N)$ the algebra, generated by invertible elements $a_i, b_i, i = 1, \dots, N$, with the following determining relations:

$$a_i b_{i-1} = q b_{i-1} a_i, \quad a_i b_i = q^{-1} b_i a_i. \quad (3.14)$$

All other generators a and b commute.

Consider the following elements in $\text{End}(C^N) \otimes A_q(N)$

$$L^+(z) = \sum_{1 \leq i \leq N} e_{i,i} \otimes a_i + \sum_{1 \leq i < N} e_{i,i+1} \otimes b_i + z^{-2} e_{N,1} \otimes b_N, \quad (3.15)$$

$$L^-(z) = \sum_{1 \leq i \leq N} e_{i,i} \otimes a_i^{-1} + \sum_{1 \leq i < N} e_{i+1,i} \otimes b_i + z^2 e_{1,N} \otimes b_N. \quad (3.16)$$

It is not difficult to check that these elements satisfy the following identities in $\text{End}(C^N) \otimes \text{End}(C^N) \otimes A_q(N)$:

$$\begin{aligned} R(x/y) L^+(x) \otimes L^+(y) &= (1 \otimes L^+(x))(L^+(y) \otimes 1) R(x/y), \\ L^-(x) \otimes L^-(y) R(x/y) &= R(x/y)(1 \otimes L^-(y))(L^-(x) \otimes 1). \end{aligned} \quad (3.17)$$

Here the tensor product is taken over the algebra $A_q(N)$ (tensor product of matrices where the elements are multiplied in $A_q(N)$). The matrix $R(z)$ is $N^2 \times N^2$ matrix acts trivially in $A_q(N)$ and is the ‘fundamental’ $U_q(\widehat{sl}(N))$ R -matrix :

$$\begin{aligned} R(z) &= (qz - q^{-1}z^{-1}) \sum_{1 \leq i \leq N} e_{i,i} \otimes e_{i,i} + (z - z^{-1}) \sum_{1 \leq i \neq j \leq N} e_{i,i} \otimes e_{j,j} \\ &+ (q - q^{-1}) \sum_{1 \leq i \neq j \leq N} (ze_{i,j} \otimes e_{j,i} + z^{-1}ze_{j,i} \otimes e_{i,j}). \end{aligned} \quad (3.18)$$

In terms of quantized universal enveloping algebras the matrices L^{\pm} describe the ‘minimal’ representations [22, 30] of the quantized Borel subalgebra $U_q(\widehat{b}_+)$ of the quantized universal enveloping algebra of $\widehat{sl}(N)$.

3.3

Here we construct integrals and the quantum Lax pair for the quantum discrete complexified Toda field theory of affine A_N type.

Consider the algebra $A_q(N, L) = A_q(N)^{\otimes 2L}$. It is generated by the elements

$$a_i(n) = 1 \otimes \dots \otimes a_i \otimes \dots \otimes 1, \quad b_i(n) = 1 \otimes \dots \otimes b_i \otimes \dots \otimes 1, \quad (3.19)$$

where $n = 1, \dots, 2L$. Define the elements $L_n^\pm(z)$ in $\text{End}(C^N) \otimes A_q(N, L)$

$$\begin{aligned} L_n^+(z) &= \sum_{1 \leq i \leq N} e_{i,i} \otimes a_i(n) + \sum_{1 \leq i < N} e_{i,i+1} \otimes b_i(n) + z^{-2} e_{N,1} \otimes b_N(n), \quad [n]_2 = 1; \\ L_n^-(z) &= \sum_{1 \leq i \leq N} e_{i,i} \otimes a_i^{-1}(n) + \sum_{1 \leq i < N} e_{i+1,i} \otimes b_i(n) + z^2 e_{1,N} \otimes b_N(n), \quad [n]_2 = 0. \end{aligned} \quad (3.20)$$

Let C^* be the group of all nonzero complex numbers with respect to the multiplication. Consider the following action of $C^{*\times 2L}$ on the algebra $A_q(N, L)$

$$\begin{aligned} a_i(n) &\mapsto \alpha_i(n) a_i(n) \alpha_i^{-1}(n-1); \\ b_i(n) &\mapsto \alpha_i(n) b_i(n) \alpha_{i+1}^{-1}(n-1), \quad [n]_2 = 1; \\ b_i(n) &\mapsto \alpha_{i+1}(n-1) b_i(n) \alpha_i^{-1}(n), \quad [n]_2 = 0. \end{aligned} \quad (3.21)$$

This action can be represented by the following ‘gauge transformation’ of matrices L_n^\pm :

$$\begin{aligned} L_n^+(z) &\mapsto D_n L_n^+(z) D_{n-1}^{-1}, \quad [n]_2 = 1; \\ L_n^-(z) &\mapsto D_{n-1} L_n^-(z) D_n^{-1}, \quad [n]_2 = 0, \end{aligned} \quad (3.22)$$

where $D_n \in \text{End}(C^N) \otimes A_q(N, L)$ act trivially in $A_q(N, L)$ and diagonally in C^N :

$$D_n = \sum_{i=1}^N \alpha_i(n) e_{ii} \otimes 1.$$

Proposition 4 *The map:*

$$\begin{aligned} \chi_i(n) &\mapsto a_{i+1}(n) b_i(n) b_i(n+1) a_i^{-1}(n+1), \quad [n]_2 = 0; \\ \chi_i(n) &\mapsto a_i(n+1) b_i(n+1) b_i(n) a_{i+1}^{-1}(n), \quad [n]_2 = 1, \end{aligned} \quad (3.23)$$

extended by linearity to the algebra $C_q(N, L)$, gives a homomorphism of algebras with the image in the subalgebra $A_q^{inv}(N, L) \subset A_q(N, L)$ of elements invariant with respect to the action of the ‘gauge group’ $C^{*\times 2L}$.

Denote the images of elements of $C_q(N, L)$ in $A_q^{inv}(N, L)$ by the same symbols. Clearly, the elements

$$\begin{aligned} A_i &\equiv \prod_{n=1}^{2L} a_i(n), \quad i = 1, \dots, N; \\ B_k &\equiv \prod_{j=1}^J b_j(2k-2j+1) b_{j+1}^{-1}(2k-2j), \quad k = 1, \dots, F, \end{aligned} \quad (3.24)$$

also belong to $A_q^{inv}(N, L)$. Their products

$$z_A \equiv \prod_{i=1}^N A_i, \quad z_B \equiv \prod_{k=1}^F B_k \quad (3.25)$$

lie in the center of $A_q(N, L)$, so we can consider a representation, where they act as complex numbers.

Proposition 5 *The subalgebra $A_q^{inv}(N, L)$ of gauge invariant elements is generated by the elements A_1 , B_1 , and $\chi_i(n)$, $i = 1, \dots, N$; $n = 1, \dots, 2L$, subject to the constraints*

$$(A_1)^N = z_A f_A(\chi), \quad (B_1)^F = z_B f_B(\chi),$$

where $f_A(\chi)$ and $f_B(\chi)$ are some monomials of their arguments.

The proof easily follows from the fact that the ratios $A_i A_{i-1}^{-1}$ and $B_k B_{k-1}^{-1}$ are monomials in elements $\chi_j(m)$.

Thus, an appropriate extension of $C_q(N, L)$ by $f_A^{1/N}(\chi)$ and $f_B^{1/F}(\chi)$ is isomorphic to $A_q^{inv}(N, L)$. Correspondingly, the evolution (2.8) can be considered in $A_q^{inv}(N, L)$.

For the description of the integrals of motion we will need also to extend somehow the evolution (2.8) to the bigger algebra $A_q(N, L)$. One way to do this is to realize the automorphisms κ_{\pm} (2.6) of the subalgebra $C_q(N, L)$ by inner ones in $A_q(N, L)$, and then, to extend the latter to the whole algebra $A_q(N, L)$. There is, however, another possibility, which provides us with the extension only up to gauge transformations in the following sense.

Consider a sequence of operators $a_i(n, t)$, $b_i(n, t)$, where the ‘time’ index takes integer values. For each fixed t these elements satisfy the defining relations of the algebra $A_q(N, L)$ (3.14), (3.19). Define the corresponding elements of $\text{End}(C^N) \otimes A_q(N, L)$ according to (3.20):

$$L_{n,t}^-(z), [n+t]_2 = 0; \quad L_{n,t}^+(z), [n+t]_2 = 1, \quad (3.26)$$

and postulate the zero curvature equation:

$$L_{n,t+1}^-(z) L_{n,t}^+(z) = L_{n-1,t+1}^+(z) L_{n-1,t}^-(z). \quad (3.27)$$

This equation is invariant with respect to the ‘time dependent’ gauge transformations:

$$\begin{aligned} L_{n,t}^+(z) &\mapsto D_{n,t} L_{n,t}^+(z) D_{n-1,t-1}^{-1}, & [n+t]_2 = 1; \\ L_{n,t}^-(z) &\mapsto D_{n-1,t} L_{n,t}^-(z) D_{n,t-1}^{-1}, & [n+t]_2 = 0, \end{aligned} \quad (3.28)$$

where

$$D_{n,t} = \sum_{i=1}^N \alpha_i(n, t) e_{ii} \otimes 1, \quad \alpha_i(n, t) \in C^*.$$

Like in usual gauge theories, equation (3.27) does not specify a unique time evolution in the algebra³ $A_q(N, L)$. The following proposition establishes a direct relationship of (3.27) with the discrete Toda field equations of motion (2.8).

Proposition 6 *Let $L_{n,t}^{\pm}(z)$, defined in (3.26), (3.20), be some solution to equation (3.27). Then, the gauge invariant (with respect to gauge transformations (3.28)) operators*

$$\begin{aligned} \chi_i(n, t) &= a_{i+1}(n, t) b_i(n, t) b_i(n+1, t) a_i^{-1}(n+1, t), & [t]_2 - [n]_2 = 1; \\ \chi_i(n, t) &= a_i(n+1, t) b_i(n+1, t) b_i(n, t) a_{i+1}^{-1}(n, t), & [n]_2 - [t]_2 = 1, \end{aligned} \quad (3.29)$$

solve the discrete Toda field equations (2.8).

³To specify the evolution in gauge theories, a ‘gauge fixing procedure’ is needed, which is not unique, see e.g. [23].

Now, we turn to the description of integrals of motion for the evolution (2.8), considered in $A_q^{inv}(N, L)$.

Consider the following ‘transfer’ matrix, element of $A_q(N, L)$:

$$t_1(z) = \text{tr}_{C^N} \left((L_{2L}^-(z))^{-1} L_{2L-1}^+(z) \cdots (L_2^-(z))^{-1} L_1^+(z) \right) \in A_q(N, L). \quad (3.30)$$

In the same way introduce the set of transfer matrices $t_l(z)$, where $l = 2, \dots, N-1$, which correspond to elements $L^{\omega_l, \pm}(z) \in \text{End}(V(\omega_l)) \otimes A_q(N)$, where $V(\omega_l)$, is the set of irreducible finite dimensional $U_q(sl(N))$ modules, associated with fundamental weights ω_l . These elements can be constructed through the fusion procedure [24] from $L^\pm(z)$.

Proposition 7 *The elements $t_l(z)$, $l = 1, \dots, N-1$, belong to $A_q^{inv}(N, L)$; commute among themselves*

$$[t_l(z), t_{l'}(z')] = 0;$$

and are conserved under the evolution (2.8), extended to $A_q^{inv}(N, L)$.

These are direct consequences of the gauge transformation law (3.22), the Yang-Baxter equations (3.17), Proposition 6, and the fusion procedure [24].

4 Discrete Toda field theory as a 3-dimentional system

Define the algebra $T_q(N, L)$, generated by invertible elements $\tau(x, y)$, $x, y \in Z$, with the following commutation relations

$$\tau(x, y)\tau(x', y') = \tau(x', y')\tau(x, y), \quad [x + y]_2 = [x' + y']_2;$$

$$\tau(x, y)\tau(x', y') = q^{G(x-x', y-y')}\tau(x', y')\tau(x, y), \quad [x' + y']_2 - [x + y]_2 = 1, \quad (4.1)$$

where

$$\begin{aligned} G(x, y) &= \frac{1}{2} \sum_{j=1}^{N/F} \left(\frac{x-y}{J} + \varepsilon_J \left(\frac{x-y-1}{2} + jL \right) \right) \\ &\times \left(\frac{x+y}{N} + \varepsilon_N \left(\frac{x+y-1}{2} + jL \right) \right), \end{aligned} \quad (4.2)$$

Here we used notations (2.1) and (2.2).

The function $G(x, y)$ satisfies the following identities:

$$G(x, y) = G(-x, -y),$$

$$G(x+1, y) + G(x-1, y) - G(x, y+1) - G(x, y-1) = 2\delta_N(y)\delta_{2L}(x-y). \quad (4.3)$$

This is the ‘non-periodic’ analog of $g(x, y)$ defined in (3.7). Their difference reads:

$$G(x, y) - g(x, y) = \frac{x+y}{2N} \varepsilon_L \left(\frac{x-y-1}{2} \right) + \frac{x-y}{2J} \varepsilon_F \left(\frac{x+y-1}{2} \right) + \frac{x^2 - y^2}{2LN}.$$

Define the elements

$$\eta(x, y) = (\tau(x-1, y)\tau(x+1, y))^{-1}\tau(x, y-1)\tau(x, y+1). \quad (4.4)$$

Proposition 8 *The elements*

$$\eta(x + 2L, y)(\eta(x, y))^{-1}, \quad \eta(x + N, y + N)(\eta(x, y))^{-1}$$

lie in the center of the algebra $T_q(N, L)$.

Using this fact, impose the periodicity conditions:

$$\eta(x + 2L, y) = \eta(x + N, y + N) = \eta(x, y). \quad (4.5)$$

Next, consider the following maps of generating elements

$$\begin{aligned} \tilde{\kappa}_{\pm}: \tau(x \mp 1, y) &\mapsto \tau(x, y), \quad [x + y]_2 = 0; \\ \tilde{\kappa}_{\pm}: \tau(x \mp 1, y) &\mapsto -(\tau(x + 1, y)\tau(x - 1, y) \\ &+ \tau(x, y + 1)\tau(x, y - 1))\tau^{-1}(x, y), \quad [x + y]_2 = 1. \end{aligned} \quad (4.6)$$

Proposition 9 *The maps $\tilde{\kappa}_{\pm}$ can be extended by linearity to algebra automorphisms*

$$\tilde{\kappa}_{\pm}: T_q(N, L) \rightarrow T_q(N, L),$$

such that

$$\tilde{\kappa} \equiv \tilde{\kappa}_+ \circ \tilde{\kappa}_- = \tilde{\kappa}_- \circ \tilde{\kappa}_+. \quad (4.7)$$

Define now the ‘time’ dependent fields:

$$\tilde{\kappa}^t(\tau(x, y)) = \begin{cases} \tau(2t, x, y), & \text{if } [x + y]_2 = 0; \\ \tau(2t - 1, x, y), & \text{otherwise.} \end{cases} \quad (4.8)$$

Notice that the field $\tau(t, x, y)$ is defined only for $[t + x + y]_2 = 0$.

Proposition 10 *The operators $\tau(t, x, y)$ satisfy the following equations*

$$\tau(t + 1, x, y)\tau(t - 1, x, y) + \tau(t, x + 1, y)\tau(t, x - 1, y) + \tau(t, x, y + 1)\tau(t, x, y - 1) = 0, \quad (4.9)$$

and permutation relations

$$\begin{aligned} \tau(t, x, y)\tau(t - 1, x', y') &= q^{G(x - x', y - y')}\tau(t - 1, x', y')\tau(t, x, y), \\ \tau(t, x, y)\tau(t, x', y') &= \tau(t, x', y')\tau(t, x, y). \end{aligned} \quad (4.10)$$

Equations (4.9) were first written by Hirota in [8] for the scalar ‘ τ -function’ $\tau(t, x, y)$, so it is natural to call them as the ‘quantum’ Hirota equations on ‘quantum τ -function’ (4.10).

Proposition 11 *There exists an algebra homomorphism*

$$\iota: C_q(N, L) \ni \chi_i(n) \mapsto -\eta(n + i, i) \in T_q(N, L),$$

such that

$$\iota \circ \tilde{\kappa}_{\pm} = \tilde{\kappa}_{\pm} \circ \iota.$$

Thus, in the ‘ τ ’-variables the discrete Toda system can be interpreted as a 3-dimensional discrete system with equations of motion (4.9), which are invariant with respect to permutations of the space-time coordinates.

5 The $N \rightarrow \infty$ limit as a 3-dimensional thermodynamical limit

Let us return to the $1 + 1$ -dimensional Toda field theory in the continuous space-time and consider it as a $2 + 1$ -dimensional field theory in partly discrete space-time: one space coordinate is discrete with the values $1, \dots, N$ (the ‘Lie algebra direction’), the other two are, as usual, continuous. We will refer to excitations in $1 + 1$ dimensional theory as one-dimensional particles and to the excitations in the $2 + 1$ -dimensional theory as two-dimensional particles.

The spectrum of the theory can be interpreted in two ways: as the spectrum of the $1 + 1$ -dimensional model and as the spectrum of the $2 + 1$ -dimensional model:

- the $1 + 1$ -dimensional interpretation: $N - 1$ massive particles with masses $M_l = M \sin(\pi l/N)$.
- the $2 + 1$ -dimensional interpretation: one scalar massless particle with the momentum in the second space direction given by the above formula.

These two interpretations are very reminiscent to the mechanism of the mass generation via the compactification of extra dimensions (see for example [25]).

As $N \rightarrow \infty$ the 2-dimensional momentum of the $2 + 1$ -dimensional scalar particle becomes continuous. This corresponds to the limit where the ratio $\pi l/N$ in the formula for the masses is kept finite as $N \rightarrow \infty$. For finite N the scattering of massive one-dimensional particles is pure elastic with the scattering amplitudes given by (1.2). The notion of scattering becomes more subtle in the limit $N \rightarrow \infty$, since we are dealing now with massless $2 + 1$ -dimensional particles. It is not difficult, however, to verify that the system has the correct 2-dimensional thermodynamical behaviour in this limit and, to compute the asymptotics of the free energy (by the 2-dimensional thermodynamical limit we mean the limit, where the 2-dimensional space volume of the system increases proportionally to the number of excitations). It is convenient to use the thermodynamical Bethe ansatz for these purposes.

5.1

The idea of using scattering amplitudes and dispersions of physical excitations for the description of states of the thermodynamical equilibrium goes back to [26]. For the Toda system this gives the following answer for the energy levels of the model in the box of length L with periodic boundary conditions (we assume that N is yet finite):

$$E = \sum_{l=1}^N \sum_{\alpha=1}^{n_l} M \sin\left(\frac{\pi l}{N}\right) \cosh \theta_{\alpha}^{(l)}, \quad (5.1)$$

where n_l is the number of particles of type l in the state and the rapidities $\theta_{\alpha}^{(l)}$ are ‘quantized’ by the periodic boundary conditions as follows:

$$LM \sin\left(\frac{\pi l}{N}\right) \sinh \theta_{\alpha}^{(l)} = 2\pi I_{\alpha}^{(l)} + \sum_{(k,\beta) \neq (l,\alpha)} \phi_{l,k}(\theta_{\alpha}^{(l)} - \theta_{\beta}^{(k)}). \quad (5.2)$$

Here the numbers $I_{\alpha}^{(l)}$ are integers and $\phi_{l,k}(\theta) = -i \ln(S_{l,k}(\theta))$ assuming that the branch of the logarithm is chosen in such a way that ϕ vanishes when $b = 0$. Similar equations have been studied in detail in [27, 28] for the chiral Gross-Neveu type models.

Since the numbers $I_{\alpha}^{(l)}$ form only a subset among all integers, one can introduce the rapidities of ‘holes’ (in the distribution of $I_{\alpha}^{(l)}$ among integers) as solutions to

the system

$$LM \sin\left(\frac{\pi l}{N}\right) \sinh \tilde{\theta}_\alpha^{(l)} = 2\pi \tilde{I}_\alpha^{(l)} + \sum_{(k,\beta)} \phi_{l,k}(\tilde{\theta}_\alpha^{(l)} - \theta_\beta^{(k)}). \quad (5.3)$$

Here $\tilde{I}_\alpha^{(l)}$ are all integers which do not belong to $\{I_\alpha^{(l)}\}$.

According to [26] we will refer to such a state as a state with particles with rapidities θ and with holes with rapidities $\tilde{\theta}$.

The 1-dimensional thermodynamical (macroscopic) states correspond to the limit $L \rightarrow \infty$ where $n_l = L\rho_l$ with finite densities ρ_l . These states are parametrized by the asymptotic densities of distributions of rapidities of particles and holes along the real line ($\rho_l(\theta)$ and $\rho_l^h(\theta)$, respectively). These densities are certainly not independent. The equations (5.2), (5.3) provide the following integral equation which relates $\rho_l(\theta)$ and $\rho_l^h(\theta)$

$$M_l \cosh \theta = 2\pi \rho_l(\theta) + 2\pi \rho_l^h(\theta) + \sum_{1 \leq k \leq N} \int_{-\infty}^{\infty} \phi'_{l,k}(\theta - \alpha) \rho_k(\alpha) d\alpha. \quad (5.4)$$

The energy of such state grows proportionally to the length of the system:

$$E = L \sum_{1 \leq l \leq N} \int_{-\infty}^{\infty} M_l \cosh \theta \rho_l(\theta) d\theta. \quad (5.5)$$

The state of the thermodynamical equilibrium minimizes the free energy of the system which is the linear combination of the energy and the entropy

$$F = E - TS. \quad (5.6)$$

Here T is the temperature and S is the combinatorial entropy of the gas of particles and holes. It has the following asymptotics as $L \rightarrow \infty$ on macroscopic states:

$$\begin{aligned} S = L \sum_{1 \leq l \leq N} \int_{-\infty}^{\infty} \{(\rho_l(\theta) + \rho_l^h(\theta)) \ln(\rho_l(\theta) + \rho_l^h(\theta)) \\ - \rho_l(\theta) \ln \rho_l(\theta) - \rho_l^h(\theta) \ln \rho_l^h(\theta)\} d\theta. \end{aligned} \quad (5.7)$$

Minimization of the functional (5.6) with the condition (5.4) gives the following formula for the free energy of the state of the thermodynamical equilibrium:

$$F = L \sum_{1 \leq k \leq N} \int_{-\infty}^{\infty} M_l \cosh \theta \ln(1 + \exp(-\epsilon_k(\theta)/T)) d\theta, \quad (5.8)$$

where the functions $\epsilon_k(\theta)$ satisfy the following system of nonlinear integral equations:

$$M_l \cosh \theta = \epsilon_l(\theta) + \frac{1}{2\pi} \sum_{1 \leq k \leq N} \int_{-\infty}^{\infty} \phi'_{l,k}(\theta - \alpha) \ln(1 + \exp(-\epsilon_k(\alpha)/T)) d\alpha. \quad (5.9)$$

5.2

Now let us consider the thermodynamical limit and thermodynamical states of the Toda field theory, regarded as a 2 + 1-dimensional model. It is not difficult to verify that the limit $N, L \rightarrow \infty$ does not depend on the order in which it is taken. Let us consider the case where we first take the limit $L \rightarrow \infty$ and then $N \rightarrow \infty$. The limit $L \rightarrow \infty$ for fixed N has been already described above. When $N \rightarrow \infty$ the 2-dimensional macroscopic states correspond to the macroscopic number of 2 + 1-dimensional excitations. This means that we have to consider the states

with $n = \sigma N$ where $n = \sum_{1 \leq l \leq N} n_l$ and σ is fixed when $N \rightarrow \infty$. The densities of holes and particles in such states will be functions of 2 variables (of 2-momentum): $\rho_l^h(\theta) \rightarrow \rho(\theta, \pi l/N)$, $\rho_l(\theta) \rightarrow \rho(\theta, \pi l/N)$.

Let $x = \pi l/N, y = \pi k/N$, and b in (1.2) be fixed, and $N \rightarrow \infty$. The function $\phi_{l,k}(\theta)$ has the following asymptotics in this limit

$$\phi_{l,k}(\theta) = \frac{8\pi^3(B^2 - B)}{N} K(\theta|x, y) + O\left(\frac{1}{N^3}\right), \quad (5.10)$$

where the $B = (1 + 4\pi/\beta^2)^{-1}$ and the function $K(\theta|x, y)$ has the following form:

$$K(\theta|x, y) = \sinh(\theta) \left\{ \frac{1}{\cosh(\theta) - \cos(x+y)} - \frac{1}{\cosh(\theta) - \cos(x-y)} \right\}. \quad (5.11)$$

Using asymptotics (5.10) in equations (5.4), (5.5), (5.7), we obtain the following description of macroscopic states in the affine Toda field theory regarded as a 2+1-dimensional field theory.

The energy and the entropy of such states are:

$$E = \frac{LN}{\pi} \int_0^\pi \int_{-\infty}^\infty M \sin(x) \cosh(\theta) \rho(\theta, x) d\theta dx, \quad (5.12)$$

$$\begin{aligned} S = & \frac{LN}{\pi} \int_0^\pi \int_{-\infty}^\infty \{(\rho(\theta, x) + \rho^h(\theta, x)) \ln(\rho(\theta, x) + \rho^h(\theta, x)) - \\ & \rho(\theta, x) \ln \rho(\theta, x) - \rho^h(\theta, x) \ln \rho^h(\theta, x)\} d\theta dx. \end{aligned} \quad (5.13)$$

The densities of holes and particles are related by the equation

$$\begin{aligned} M \sin x \cosh \theta = & 2\pi \rho(\theta, x) + 2\pi \rho^h(\theta, x) + 8\pi^4(B^2 - B) \\ & \times \int_0^\pi \int_{-\infty}^\infty K'(\theta - \alpha|x, y) \rho(\alpha, x) d\alpha dy. \end{aligned} \quad (5.14)$$

Minimizing the free energy (5.6), we obtain the following expression for the free energy of the Toda model for large N :

$$F(T) = \frac{LN}{\pi} \int_0^\pi \int_{-\infty}^\infty M \sin(x) \cosh \theta \ln(1 + \exp(-\epsilon(\theta, x)/T)) d\theta dx, \quad (5.15)$$

where the function $\epsilon(\theta, x)$ is the solution to the following nonlinear integral equation:

$$\begin{aligned} M \sin x \cosh \theta = & \epsilon(\theta, x) + 4\pi^3(B^2 - B) \\ & \times \int_0^\pi \int_{-\infty}^\infty K'(\theta - \alpha|x, y) \ln(1 + \exp(-\epsilon(\alpha, y)/T)) d\alpha dy. \end{aligned} \quad (5.16)$$

These equations describe the equilibrium thermodynamics of the Toda system at $N \rightarrow \infty$.

6 Conclusion

In this paper we studied the Toda field theory along two lines. We investigated the two dimensional thermodynamical limit of this model, and constructed the discrete space-time approximation which partly has the discrete Lorentz invariance.

6.1

Let us analyze the continuum limit in the Toda field theory, where the third dimension becomes continuous as well. It is not difficult to see from the Lagrangian (1.1) that such limit corresponds to $\beta \rightarrow \infty$ and $M = m\beta$ with m being fixed. As a result we have the theory in $2 + 1$ -dimensional space-time with the Lagrangian

$$L_{AT} = \frac{1}{2} \int \int \left(\left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 - 2m^2 \exp \left(\frac{\partial \phi}{\partial y} \right) \right) dx dy. \quad (6.1)$$

Equations (5.15) and (5.16) imply the following asymptotics of the free energy in this limit:

$$F(T) = LNT \int_0^\pi \int_{-\infty}^\infty mt \cosh(\theta) \ln(1 + \exp(-mt \cosh(\theta)/T)) d\theta dt.$$

This is the free energy of massless free particles. Such a behaviour of the free energy suggests that in the continuum limit the Toda system describes noninteracting particles. From the structure of the asymptotics of the free energy one can assume that the particles are fermions.

The fact that in the continuum limit the Toda field theory describes noninteracting free particles also can be seen from the corresponding limit in the Bethe equations (5.2). Recall that these equations describe possible values of rapidities of physical particles in the box of length L with periodic boundary conditions. In the continuum limit $\beta \rightarrow \infty$, $M = m\beta$, $N = L_1\beta$ with fixed m and L_1 the equations (5.2) degenerate into the equations

$$Lp = 2\pi I, \quad L_1 p_1 = \pi l,$$

where $p = mp_1 \sinh(\theta)$. The energy of such excitation, according to (5.1), is

$$E^2 = p^2 + p_1^2.$$

It is clear that this is the spectrum of free relativistic massless two dimensional particles. Thus, the Toda theory has some selfinteraction for finite β and $N \rightarrow \infty$ but it becomes free in the continuum limit.

6.2

The large N limit in the principal chiral field theory, based on the group $SU(N)$, has been studied in the work [29]. The difference between this model and the Toda theory is obvious: the large N limit of the principal chiral field theory describes some string-type objects, while the similar limit in the Toda field theory describes the $2 + 1$ -field theory.

Now let us conclude with some open problems and conjectures.

- One has to understand the relation between the 3-dimensional model constructed in [31], its two dimensional counterpart [30], and the quantum discrete system constructed in sections 2-4. We conjecture that the model constructed in [31] corresponds to the discrete quantum Toda model at roots of 1. The relation should be similar to the one between the discrete sine-Gordon and the chiral Potts model [32].
- When q is a root of 1, the discrete Toda system has properties similar to the discrete sine-Gordon system at roots of 1: it describes the quantum integrable system, interacting with the classical integrable system.

- It is interesting to compute the spectrum of the discrete quantum Toda field theory. By the analogy with the continuum model and with the discrete sine-Gordon system, we conjecture the following spectrum of the model in the infinite interval along the x -coordinate. Let T_x, T_t be the translation operators in x and t directions, respectively:

$$T_x a_{n,t} T_x^{-1} = a_{n+2,t}, \quad T_t a_{n,t} T_t^{-1} = a_{n,t+2}.$$

The common eigenstates of the Hamiltonians, produced by the generating function (3.30) and of the operators T_x, T_t , form a Fock space with the same structure of particles as in the continuum case, and with the same scattering amplitudes. The translation operators T_x, T_t have the following eigenvalues on the state with one particle of type l with the rapidity θ :

$$T_x |\theta\rangle = \frac{\cosh(\frac{\theta}{2} + \Lambda + \frac{i\pi l}{2N}) \cosh(\frac{\theta}{2} - \Lambda + \frac{i\pi l}{2N})}{\cosh(\frac{\theta}{2} + \Lambda - \frac{i\pi l}{2N}) \cosh(\frac{\theta}{2} - \Lambda - \frac{i\pi l}{2N})} |\theta\rangle,$$

$$T_t |\theta\rangle = \frac{\cosh(\frac{\theta}{2} + \Lambda + \frac{i\pi l}{2N}) \cosh(\frac{\theta}{2} - \Lambda - \frac{i\pi l}{2N})}{\cosh(\frac{\theta}{2} + \Lambda - \frac{i\pi l}{2N}) \cosh(\frac{\theta}{2} + \Lambda - \frac{i\pi l}{2N})} |\theta\rangle.$$

The eigenvalues of these operators on many particle states are products over the individual particles of one particle contributions.

- The difference between the large N limits in the Toda field theories, related to other classical Lie algebras and the $SL(N)$, can be interpreted as the other (nonperiodic) boundary conditions in the extra dimension. It would be interesting to compute the corresponding bulk terms in the free energy.
- We have constructed the quantum analog of the τ -functions, introduced and studied in [8, 11]. It would be interesting to obtain the formulas for these quantum τ -functions which would generalize the determinant formulas or similar constructions, known in the classical case.

We are planing to return to these problems in the extended version of this publication.

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